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Large Amplitude Linear Vibrations of Tensioned Strings

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ABSTRACT

The transverse motion of a tensioned string driven parametrically by a moving end support is examined both analytically and experimentally. The conditions required to linearize the equations of motion for vibrations over a large amplitude range at constant frequency are developed and shown to be physically realizable.

INTRODUCTION

The transverse vibrations of a tensioned string mounted between two rigid end supports is a classic problem when the amplitude of vibration is small and the initial tension in the string is unaltered by its motion. Several analyses have been made of the nonlinear regime encountered when

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the motion of the tensioned string is large enough to influence the tension.^{1,2,3,4} There has been recent interest in the application of the vibrating string to various measurement and sensing problems, for example, measuring lineal motions along or perpendicular to the rest axis of the string and measuring angular motion about its rest axis. If small amplitudes are used in such applications, linear equations describe the string motion but complex measuring apparatus must be used. Large amplitude nonlinear vibration, on the other hand, is characterized by complex drive requirements (imposed by stability considerations) and inherent losses in accuracy and reliability associated with these added complexities.

Intuitively it appears possible to obtain large amplitudes with essentially no change in natural frequency with amplitude if the supports are moved in a manner to offset increased tension due to deflection. With this in mind we will investigate the large amplitude vibration of a string under approximately constant tension for the purpose of demonstrating the feasibility of its utilization as a linear element in sensing systems.

MOTION OF A STRING UNDER APPROXIMATELY CONSTANT TENSION

Let us consider a string under a tension $T = T_0 + \Delta T(s,t)$ where $\Delta T(s,t) \ll T_0$ and let us allow it to vibrate so that it has a constant average tension equal to T_0 . The relative significance of this choice of operation will be apparent later. Since this condition might be achieved, for example, by appropriate control of the axial motion of one of the end supports, let us initially assume that the string is mounted (as shown in Fig. 1) between two rigid end supports, A and B, with A taken as the fixed reference for our coordinate system, and B allowed to have motion along the rest axis of the string. Additional assumptions will be those normally invoked:

Fig. 1

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- a) The propagation time along the string is small compared to the period of transverse vibration.
- b) The forces introduced by the bending of the string are negligible.
- c) The motion of the string is confined to the x-y plane.
- d) The damping forces are negligible.

The partial differential equations describing the motion of the string are:

$$m \left(\frac{\partial^2 x}{\partial t^2} \right) = \frac{\partial [T(\partial x / \partial s)]}{\partial s} \quad (1)$$

$$m \left(\frac{\partial^2 y}{\partial t^2} \right) = \frac{\partial [T(\partial y / \partial s)]}{\partial s} \quad (2)$$

For the moment we will neglect the effect of the x-directed accelerations on the tension distributed along the string. In the classical solution of Eq. (2) ds is assumed equal to dx . For large deflections, however, this assumption is unwarranted⁵ and will not be made at this time. Considering only the steady-state solution for string motion in the s-y plane and in the fundamental mode, the well-known solution for Eq. (2) is

$$y(s,t) = y_0 \sin(\pi s/s_0) \sin \omega_0 t \quad (3)$$

where ω_0 is the low amplitude natural frequency of the string.

One important assumption made in reaching this solution was that the tension remains constant. This assumption implies that the motion of the support block B would be so constrained as to hold the instantaneous value of the tension, measured along the string, equal to the tension, T_0 . Whether or not this condition can be achieved is the next question to examine.

Let us note first that from purely geometrical considerations

$$(dx)^2 = (ds)^2 [1 - (\partial y / \partial s)^2]$$

The partial derivative of y with respect to s can be obtained from Eq. (3) and inserted in the above equation to provide

$$(dx)^2 = (ds)^2 [1 - (\pi^2 y_0^2 / s_0^2) \sin^2 \omega_0 t \cos^2(\pi s / s_0)] \quad (4)$$

Taking the square root of both sides of this equation and making the series approximation for the square root of the quantity in brackets, one obtains the approximate expression

$$dx \approx [1 - (\pi^2 y_0^2 / 2s_0^2) \sin^2 \omega_0 t \cos^2(\pi s / s_0)] ds$$

and integrating this equation gives

$$x = s - (\pi^2 y_0^2 / 4s_0^2) s \sin^2 \omega_0 t - (\pi y_0^2 / 8s_0) \sin(2\pi s / s_0) \sin^2 \omega_0 t \quad (5)$$

Evaluating Eq. (5) at the point $s = s_0$, we obtain the expression

$$x_D = s_0 - (\pi^2 y_0^2 / 4s_0) \sin^2 \omega_0 t \quad (6)$$

which describes the motion of the end block required to maintain a constant tension.

The solution of the partial differential equation in y has produced frequency and shape equations for the motion of the string in the s - y plane,⁶ and has provided the equation of motion of the end support required to maintain a constant average tension. It has still required the assumption that tension is uniform along the string. In examining the validity of this assumption, let us assume that x accelerations are produced by a combination of two forces: first, the x -directed force resulting from the constant tension component and the curvature of the string; second, the additional x -directed force (not accounted for by the component introduced by the curvature and the initial tension) required to

satisfy the shape and frequency equations already derived. This latter component will result in a time varying tension distribution along the string. If we derive the expression describing this distribution as a function of the deflection, y_0 , and compare the magnitude of the tension variations with the magnitude of the initial tension, this comparison will allow us to judge the validity of the assumption that forces introduced by x accelerations are negligible and thereby to determine the deflection region for which this assumption is valid.

The forces acting in the x direction are described by Eq. (1)

$$m \frac{\partial^2 x}{\partial t^2} ds = \frac{\partial}{\partial s} \left(T \frac{\partial x}{\partial s} \right) ds$$

where

$$T = T_0 + \Delta T(s, t)$$

If we solve the above equation for T using the expression for x given by Eq. (5), we obtain

$$\frac{T}{T_0} = K(t) + \frac{\pi^2 y_0^2}{8 s_0^2} \left(1 + \cos \frac{2\pi s}{s_0} - \cos 2\omega_0 t - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t \right) \quad (7)$$

Since the initial formulation requires that the instantaneous value of the tension averaged over the length of the string be constant and equal to T_0 , we may write

$$\frac{1}{s_0} \int_0^{s_0} \frac{T}{T_0} ds = 1$$

Substituting Eq. (7) for T/T_0 , integrating, and solving for the constant of integration, we obtain

$$K(t) = 1 + \frac{\pi^2 y_0^2}{8s_0^2} \left(\cos 2\omega_0 t + \frac{2\pi^2}{3} \cos 2\omega_0 t - 1 \right)$$

and, hence,

$$\frac{T}{T_0} = 1 + \frac{\pi^2 y_0^2}{8s_0^2} \left(\cos \frac{2\pi x}{s_0} + \frac{2\pi^2}{3} \cos 2\omega_0 t - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t \right) \quad (8)$$

From this equation we obtain a maximum value for tension variation of

$$\left(\frac{\Delta T}{T_0} \right)_{\max} = \frac{\pi^2 y_0^2}{8s_0^2} \left(1 - \frac{2\pi^2}{3} + 2\pi^2 \right) \approx 17.5 \frac{y_0^2}{s_0^2} \quad \text{at } \omega_0 t = \frac{\pi}{2}$$

Thus, when the average tension of the string is maintained instantaneously equal to T_0 and an instantaneous maximum local tension variation of $(\Delta T/T_0)_{\max} = 0.05$ is tolerable, the corresponding range of y_0 for which Eq. (3) is valid is $y_0/s_0 \leq 0.055$.

For purposes of this paper, we are primarily concerned with the characteristics of a string driven by the motion of an end support so as to maintain its average tension constant. It is interesting to note, however, that the approach used above can be applied to the string vibrating between fixed end supports to yield a solution which should apply over a larger range of vibration amplitudes than those derived by the classical analysis.⁸

MOTION OF THE STRING FOR $\sin^2 \omega t$ END DRIVE

The analysis presented above has not required the basic assumption that $ds = dx$ nor the implied assumption that an element of the string must remain in the vertical interval between x and $x + \Delta x$, but it has been limited to $\omega = \omega_0$. A more complete understanding of the string motion requires examination of its motion for $\omega \neq \omega_0$ and its stability.

Previous analyses of string motion have been made in the x - y coordinate system. In order that the results of our analysis of motion at frequencies other than ω_0 and our analysis of stability can be more readily

compared with other analyses, it is convenient at this point to treat these problems in the more conventional coordinates, making the small amplitude approximations and assuming that ignoring the effect of the moving boundary does not significantly affect the results. The resulting equation for the deflection of the string for the constant tension case is then

$$y(x,t) = y_0 \sin \frac{\pi x}{s_0} \sin \omega_0 t$$

It is clear that, if the values of y_0/s_0 and s are sufficiently restricted, the displacements of an element of the string obtained from both the s coordinate analysis and the x coordinate analysis are essentially equal. Being cognizant of this comparison for the special case where $\omega = \omega_0$ we will examine in a conventional way the motion of a vibrating string driven by an axial end motion, $-x_d \sin^2 \omega t$, for the case where tension is uniform along the string but is allowed to vary with time, and ω is allowed to assume values other than ω_0 . The frequency response and stability criteria resulting from this analysis will be checked experimentally.

The motion of a string under initial tension, T_0 , supported between one stationary end support and a second end support moving with an amplitude $x_d' \cos 2\omega t$ (Fig. 1) has been considered by Quick.¹ The equation of motion for such a string is

$$m \frac{\partial^2 y}{\partial t^2} = \left[T_0 + \frac{E a x_d'}{s_0} \cos 2\omega t + \frac{E a}{s_0} \int_0^{s_0} \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2 dx \right] \frac{\partial^2 y}{\partial x^2} \quad (9)$$

where m is the mass per unit length, E is Young's modulus, and a is the cross-sectional area of the string. The derivation of this equation requires that the amplitude of vibration be sufficiently small that the sine and tangent of the deflection angle can be assumed equal. The terms in brackets in the above equation represent the initial tension, the tension due to driver motion, and the tension due to change in arc length, respectively. For the term involving the arc length, the approximation has been made that

$$\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \approx 1 + \frac{1}{2} \left(\frac{\partial y}{\partial x}\right)^2$$

The propagation time along the string is assumed to be small compared to the period of transverse vibration, and the forces introduced by the bending of the string are assumed negligible. Only planar motion of the string is considered.

The motion of a string driven by the axial motion of an end support moving with an amplitude $-x_d \sin^2 \omega t$ can be analyzed in the same manner. The equation of motion for this condition is:

$$m \frac{\partial^2 y}{\partial t^2} = \left[T_0 - \frac{Eax_d}{s_0} \sin^2 \omega t + \frac{Ea}{s_0} \int_0^{s_0} \frac{1}{2} \left(\frac{\partial y}{\partial x}\right)^2 dx \right] \frac{\partial^2 y}{\partial x^2} \quad (10)$$

The trigonometric identity $\sin^2 \omega t = \frac{1}{2} - \frac{1}{2} \cos 2\omega t$ can be used to write Eq. (10) as:

$$m \frac{\partial^2 y}{\partial t^2} = \left[\left(T_0 - \frac{Eax_d}{2s_0} \right) + \frac{Eax_d}{2s_0} \cos 2\omega t + \frac{Ea}{s_0} \int_0^{s_0} \frac{1}{2} \left(\frac{\partial y}{\partial x}\right)^2 dx \right] \frac{\partial^2 y}{\partial x^2} \quad (11)$$

This equation is identical in form to Eq. (9) and has the same boundary conditions, $y(0,t) = 0$, $y(s_0,t) = 0$. It can be solved by the technique of separation of variables as outlined below.

If $y(x,t) = F(x) G(t)$, the resulting ordinary differential equation for $F(x)$ is

$$\frac{d^2 F(x)}{dx^2} + \nu_n^2 F(x) = 0 \quad (12)$$

The solution to this equation is

$$F(x) = A_n \sin \nu_n x + B_n \cos \nu_n x \quad (13)$$

In order to satisfy the boundary conditions it is necessary that $B_n = 0$ and $\nu_n = (\pi n/s_0)$ where n is a positive integer. Considering the steady-state

solution for the fundamental mode, Eq. (13) becomes

$$F(x) = A_1 \sin(\pi x/s_0) \quad (14)$$

The time-dependent equation is

$$\frac{d^2 G(t)}{dt^2} + \left[\omega_0^2 \left(1 - \frac{x_d}{2\Delta L_0} \right) + \frac{\omega_0^2 x_d}{2\Delta L_0} \cos 2\omega_0 t \right] G(t) + \frac{\pi^2 a_1^2 \omega_0^2 G^3(t)}{4s_0 \Delta L_0} = 0 \quad (15)$$

where $\Delta L_0/s_0$ is the initial strain and

$$\omega_0 = \frac{\pi}{s_0} \sqrt{\frac{T_0}{m}}$$

A particular solution for this equation can be obtained if an approximate solution of the form $G(t) = K_1 \sin \omega t + K_3 \sin 3\omega t$ is assumed. When this solution is substituted into Eq. (15), the resulting expression consists of terms in $\sin(2n+1)\omega t$; $n = 0, 1, 2, \dots$. If the coefficients of the $\sin \omega t$ and $\sin 3\omega t$ terms are equated to zero, the following relationships are obtained

$$\left. \begin{aligned} y_1^2 = K_1^2 a_1^2 &= \left[\frac{4s_0 x_d}{\pi^2} + \frac{16(\omega^2/\omega_0^2 - 1)s_0 \Delta L_0}{3\pi^2} \right] \\ \frac{y_3}{y_1} = \frac{K_3 a_1}{K_1 a_1} &= \left\{ \frac{\omega_0^2 - \omega^2}{3[7\omega^2 + \omega_0^2 - \omega_0^2(x_d/\Delta L_0)]} \right\} \end{aligned} \right\} \quad (16)$$

Thus the steady-state solution to Eq. (11) is reasonably well approximated by

$$y(x,t) = (y_1 \sin \omega t + y_3 \sin 3\omega t) \sin \frac{\pi x}{s_0} \quad (17)$$

It may be noted that the $3\omega t$ frequency term disappears if $\omega = \omega_0$ regardless of the drive amplitude. This condition also results in the tension being constant at the value T_0 .

STABILITY

In order to determine whether the motion of the string described by Eq. (15) is stable, it is necessary to examine the effect of applying a perturbation to the particular solution. The effect of the perturbation will die out if the solution is stable. To be more explicit, we will substitute the function $G(t) = G_p(t) + \alpha(t)$ into Eq. (15) and show that the resulting equation is identical in form with the Mathieu equation which has a well-established criterion for stability. Making this substitution for $G(t)$ in Eq. (15) and neglecting terms in α^2 and α^3 one obtains the resulting equation

$$\frac{d^2\alpha}{dt^2} + \frac{\omega_0^2}{\omega^2} \left[\left(1 - \frac{x_d}{2\Delta L_0} \right) \omega^2 + \frac{x_d}{2\Delta L_0} \omega^2 \cos 2\omega t + \frac{3\pi^2 \omega^2 a_1^2 G_p^2}{4s_0 \Delta L_0} \right] \alpha = 0 \quad (18)$$

Since $K_3 \ll K_1$

$$G_p^2 \approx K_1^2 \sin^2 \omega t$$

Eq. (18) becomes

$$\frac{d^2\alpha}{dt^2} + \frac{\omega_0^2}{\omega^2} \left[\left(1 - \frac{x_d}{2\Delta L_0} + \frac{3}{8} \frac{\pi^2 y_1^2}{s_0 \Delta L_0} \right) \omega^2 + \left(\frac{x_d}{2\Delta L_0} - \frac{3}{8} \frac{\pi^2 y_1^2}{s_0 \Delta L_0} \right) \omega^2 \cos 2\omega t \right] \alpha = 0 \quad (19)$$

The Mathieu equation written in the standard form is

$$\frac{d^2 z}{dt^2} + (\omega_p^2 + 16\omega_q^2 \cos 2\omega t) z = 0 \quad (20)$$

It is evident that this equation is identical to Eq. (19) if

$$p = \frac{\omega_0^2}{\omega^2} \left(1 - \frac{x_d}{2\Delta L_0} + \frac{3}{8} \frac{\pi^2 y_1^2}{s_0 \Delta L_0} \right)$$

$$q = \frac{1}{16} \frac{\omega_0^2}{\omega^2} \left(\frac{x_d}{2\Delta L_0} - \frac{3}{8} \frac{\pi^2 y_1^2}{s_0 \Delta L_0} \right)$$

The oscillatory motion described by this equation is stable if any of the conditions $p > (1+8q)$, $p < (1-8q)$ for $q > 0$ or $p < (1+8q)$, $p > (1-8q)$ for $q < 0$ are satisfied. When these criteria are applied to Eq. (19), stable oscillations can be obtained if either

$$\frac{x_d}{\Delta L_0} > \frac{4}{3} \left(1 - \frac{\omega^2}{\omega_0^2} \right) \quad \text{for } q > 0$$

or

$$\frac{x_d}{\Delta L_0} > 0 \quad \text{for } q < 0$$

Figure 2 is a graphic representation of the stability criteria for the vibrating string driven with a $-x_d \sin^2 \omega t$ forcing function. An examination of Fig. 2 shows that the oscillations are stable for any amplitude.

Fig. 2

Equation (15) can also be examined to establish the conditions necessary to start oscillations by means of a $\sin^2 \omega t$ end drive. For very small amplitudes, the cubic term can be neglected and Eq. (15) becomes

$$\frac{d^2 G(t)}{dt^2} + \frac{\omega_0^2}{\omega^2} \left[\left(1 - \frac{x_d}{2\Delta L_0} \right) \omega^2 + \frac{x_d}{2\Delta L_0} \omega^2 \cos 2\omega t \right] G(t) = 0 \quad (21)$$

We are now interested in instability; that is, conditions that make small oscillations grow. From Eq. (20), the criterion for instability is

$1 - 8q < p < 1 + 8q$. Expressing this condition in the parameters of Eq. (21) gives the relationship

$$\frac{4}{3} \left(1 - \frac{\omega^2}{\omega_0^2} \right) < \frac{x_d}{\Delta L_0} < 4 \left(1 - \frac{\omega^2}{\omega_0^2} \right)$$

which must be satisfied to start oscillations. This criterion is shown in Fig. 2.

It can be concluded from an examination of the stability criteria that the string parametrically driven by a $\sin^2 \omega t$ end motion will vibrate with stability at large amplitudes over a wide frequency spectrum. By contrast, large amplitude motion of strings driven by transverse forcing functions are characterized by unstable behavior, such as the "jump phenomenon" (sudden shifts from one mode of vibration to another at a given frequency).

For purposes of comparison it is interesting to consider the equations as obtained by Quick¹ for the displacement of a string driven by an end motion of the form $x_d' \cos 2\omega t$. The displacement is

$$y(x, t) = [(y_1 \sin \omega t + y_3 \sin 3\omega t)] \sin(\pi x / s_0) \quad (22)$$

where

$$\left. \begin{aligned} y_1^2 &= \frac{8s_0}{3\pi^2} \left[x_d' + 2\Delta L_0 \left(\frac{\omega^2}{\omega_0^2} - 1 \right) \right] \\ \frac{y_3}{y_1} &= \frac{[(\omega^2/\omega_0^2 - 1)] - (x_d'/\Delta L_0)}{3[(x_d'/\Delta L_0) - 7(\omega^2/\omega_0^2 - 1)]} \end{aligned} \right\} \quad (23)$$

The stability criteria which must be satisfied are

$$\frac{x_d'}{\Delta L_0} > 2 \left(1 - \frac{\omega^2}{\omega_0^2} \right) \quad \text{for } \frac{\omega^2}{\omega_0^2} < 1$$

and

$$\frac{x_d'}{\Delta L_0} > 0 \quad \text{for } \frac{\omega^2}{\omega_0^2} > 1$$

The initial start conditions are

$$\frac{x_d'}{\Delta L_0} > 2 \left(1 - \frac{\omega^2}{\omega_0^2} \right)$$

and

$$\frac{x_d'}{\Delta L_0} > 2 \left(\frac{\omega^2}{\omega_0^2} - 1 \right)$$

The condition necessary for linearizing the equation of motion and maintaining constant tension (i.e., no dynamic component of tension, but tension will not be equal to T_0) for this forcing function as obtained from Eq. (23) is

$$\frac{x_d'}{\Delta L_0} = \frac{\omega^2}{\omega_0^2} - 1$$

This condition indicates that, unlike the case of the $\sin^2 \omega t$ drive, the string cannot be operated linearly at its low-amplitude natural frequency, ω_0 , but must be operated at a frequency governed by the amplitude. Despite the coupling of frequency and amplitude, it is important to note that the response of an instrument using a string driven either as suggested by Quick¹ or as suggested by this paper can be linearized by the use of appropriate electronic signal conditioners. When the string is driven by an axial support motion of the form $x_d' \cos 2\omega t$, however, the input command signal must control both a change of driver amplitude in accordance with the expression

$x_d' = \pi^2 y_1^2 / 8s_0$ and a simultaneous change in frequency in accordance with the expression $\omega^2 = \omega_0^2 [(x_d' / \Delta L_0) + 1]$. In the case of the string driven as suggested by this paper, the input must control the driver amplitude in accordance with the expression $x_d = \pi^2 y_0^2 / 4s_0$ while the frequency can be allowed to remain fixed at ω_0 , independent of drive amplitude. The simplification of the system thus achieved is considerable.

It is of further interest to compare different methods of driving the string on the basis of the time-averaged strain induced in the string as a function of vibration amplitude. Without deriving the exact frequency relationships involved, it is apparent that, since the change in strain is related to the change in frequency, it provides a measure of the system nonlinearity. For the drive method suggested by this paper (the $\sin^2 \omega_0 t$ drive) the strain is forced to remain zero. For a fixed end support system, where the string is driven transversely, the change in strain is given by $\Delta x / s_0 = \pi^2 y_0^2 / 4s_0^2$. For linear operation with the $\cos 2\omega t$ end drive, the change in strain is given by $\Delta x / s_0 = \pi^2 y_0^2 / 8s_0^2$.

EXPERIMENTAL PROCEDURE

The experimental apparatus used to investigate the motion of the string is shown in Fig. 3. An electromagnet having an AC coil and a DC coil was mounted at one end of a rigid frame. The moving end support of the string was attached to the coil form which was attached to the frame by two sets

Fig. 3

of flexures. The string amplitude, y_0 , was measured both optically and by means of a displacement transducer which measured the capacitance change between the moving string and a fixed plate. The motion of the moving end support was determined from the capacitance change between a fixed plate and a plate attached to the moving end support. The principal resonance of the electromagnetic driver was approximately 5 cps. Secondary resonances of the support structure were observed at other frequencies. Data were not taken at these frequencies in order to avoid spurious effects of uncontrolled forcing functions. The AC coil was driven by an audio oscillator and power amplifier while the current in the DC coil was adjusted in the following manner: The spacing between the plate was measured with the string at rest. The string was then set in motion by applying a current $I_1 \cos 2\omega t$ to the AC coil, and the amplitude of the corresponding dynamic motion, $A \cos 2\omega t$, was measured. A current was then applied to the DC coil and adjusted so that the rest position of the end support was displaced inwardly an amount A . The net inward motion of the end support was then $A - A \cos 2\omega t = x_d \sin^2 \omega t$.

RESULTS

The amplitudes which could be examined experimentally were restricted by end driver resonances and power limitations to values of y_0/s_0 less than 0.0175 (corresponding to peak-to-peak displacements less than 1.4 cm), so that the full range of displacement amplitudes considered in the theory could not be explored. The results of tests performed with a 0.0076 cm diameter nickel steel wire 40 cm long are presented in Figs. 4 and 5. Figure 4 shows y_0 , the maximum transverse amplitude, plotted as a function of ω/ω_0 for various values of x_d where the end driver motion is

Figs. 4
and 5

$-x_d \sin^2 \omega t$. The theoretical values of y_0 obtained from Eq. (16) are represented by the solid curves. Figure 5 is a similar plot for the same wire with a driver motion $x_d' \cos 2\omega t$. An approximate value for the low-amplitude natural frequency, ω_0 , was obtained from the initial start behavior (i.e., ω_0 was taken to be that frequency at which vibration could be most easily initiated). A better value for ω_0 was then determined by keeping frequency constant and vibrating the wire at several different amplitudes. Equation (18) was then solved for ω_0 , and this value was used for all other calculations related to this initial tension. Figures 2 and 6 show the regions where vibrations can be initiated and the regions in which stable oscillations exist for driver motions of $\sin^2 \omega t$ and $\cos 2\omega t$, respectively. The experimental points for "start up" and "drop out" are also plotted on these figures.

The experimental results show that the string motion described by the analysis correlates with its measured motion both in a vacuum and in the presence of air damping.⁹ For frequencies other than ω_0 the string motions obtained by the $\sin^2 \omega t$ drive and the $\cos 2\omega t$ drive are essentially equivalent. It can be seen from the dotted line of Fig. 4, however, that $\omega = \omega_0$ is precisely the condition required to linearize the equations of motion of the string forced by a $\sin^2 \omega t$ end drive. By contrast, the conditions required to linearize the equation of motion of the string forced by a $\cos 2\omega t$ end drive (see dotted line of Fig. 5) involve correlating small changes in amplitude with large changes in frequency. Operation of either system at conditions other than those shown by the dotted line would involve large cyclic changes in tension, limited operating lifetime due to fatigue caused by those changes, and would require a significantly more complex control system to stabilize the nonlinear behavior of the string.

Fig. 6

Clearly then, the proposed $\sin^2 \omega t$ end drive offers significant advantages.

CONCLUDING REMARKS

The equation of motion of a vibrating string and the form of the parametric drive required to obtain essentially constant tension and therefore a linear equation have been derived. By the use of s - y coordinates, an analysis of the motion has been made, which is not entirely rigorous but has enabled us to judge the amplitude range for which the constant tension assumption is valid, and obtain a first approximation of the function which describes the small deviation of tension from the assumed constant value. Several methods of physically controlling the tension have been shown to produce essentially equivalent results. Results of a less precise (x, t) analysis (used because of their canonical form) then allowed us to study stability criteria. Experimental results over moderate amplitude ranges show good correlation with the theory, and further indicate that the effect of air damping (ignored in the derivation of the equations of motion of the string) is negligible. The analysis is shown to apply, therefore, to large amplitude stable vibrations with damping.

FOOTNOTES

¹W. H. Quick and G. A. Barnes, EM-262-482, ASD-TDR-62-898.

²N. W. McLachlan, Theory of Vibrations (Dover, New York, 1951).

³D. W. Oplinger, J. Acoust. Soc. Am. 32, 12 (1960).

⁴G. F. Carrier, Quart. Appl. Math. 3, 157-165 (1945).

⁵Using this assumption to derive the differential equation generally involves summing the y -directed forces acting on a particular string element and assuming implicitly that the element whose motion is thus described remains confined between x and $x + dx$. It is clear, however, that these conditions are not physically valid. To solve the problem more rigorously, one must develop the equations that describe the composite motion of an element moving in an arc; or, alternatively, derive the equations which describe the motion of the varying element of the string contained between x and $x + dx$. Eliminating this assumption introduces mathematical complexities which may be avoided if the transverse deflection, y , is analyzed as a function of (s) and (t) instead of the normal variables (x) and (t) . Further, the use of the s - y coordinate system eliminates the need for working with a boundary value problem where one of the boundaries (the moving end support) is in motion in the x direction. After the partial differential equation relating y , s , and t is solved, $y(x,t)$ can be derived.

⁶For the sake of completeness, it would now be desirable to use Eqs. (3) and (4) to derive $y(x,t)$, describing the motion in a more familiar frame of reference. Integrating Eq. (4) to determine $s = f(x,t)$, however, involves the inverse of the function $E[(\pi s/s_0) - (\pi/2), (\pi y/s_0) \sin \omega_0 t]$ where E is an elliptic integral of the second kind. It is apparent, then, that $y(x,t)$ cannot be expressed in terms of the simple algebraic, circular or hyperbolic

functions of elementary mathematics. An approximate solution for $y(x,t)$ can be obtained from Eq. (5). This solution is

$$y(x,t) = y_0 \sin \omega_0 t \sin \frac{\pi x}{s_0} \left[1 + \frac{\pi^3 y_0^2 x}{4 s_0^3} \cot \frac{\pi x}{s_0} \sin^2 \omega_0 t \left(1 + \frac{\sin 2\pi x/s_0}{2\pi x/s_0} \right) \right]$$

⁷This condition has the desired characteristic that the length of the string is then unchanging regardless of its modulus of elasticity. However, it is worthwhile to consider several other operating conditions which still meet the condition that $\Delta T \ll T_0$ and examine their associated tension deviations. While the latter cases do not hold the average tension quite constant and therefore allow some overall stretching or relaxation of the string, they may be easier to implement for some applications. A few of the many possible alternative conditions are outlined below:

When T is instantaneously equal to T_0 at $s = 0$

$$\frac{T}{T_0} = 1 + \frac{\pi^2 y_0^2}{8 s_0^2} \left(\cos \frac{2\pi s}{s_0} - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t - 1 \right)$$

and

$$\left(\frac{\Delta T}{T_0} \right)_{\max} \approx 24 \frac{y_0^2}{s_0^2}$$

When a constant force is applied by the string on the driven support, that is,

$$T(\partial x / \partial s) = T_0$$

$$\frac{T}{T_0} = 1 + \frac{\pi^2 y_0^2}{8 s_0^2} \left(\cos \frac{2\pi s}{s_0} - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t + 2\pi^2 \cos 2\omega_0 t - 2 \cos 2\omega_0 t + 1 \right)$$

and

$$\left(\frac{\Delta T}{T_0} \right)_{\max} \approx 24 \frac{y_0^2}{s_0^2}$$

When T is instantaneously maintained equal to T_0 at $s = s_0$

$$\frac{T}{T_0} = 1 - \frac{\pi^2 y_0^2}{8s_0^2} \left(1 - \cos \frac{2\pi s}{s_0} - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t + 2\pi^2 \cos 2\omega_0 t \right)$$

and

$$\left(\frac{\Delta T}{T_0} \right)_{\max} \approx 24 \frac{y_0^2}{s_0^2}$$

In the discussion above one support has been fixed while the other has been constrained to follow the x component of the string motion. An alternative physical arrangement might allow no x motion at the center while constraining both supports to follow the x motion of the string. When the tension averaged along the length of the string is instantaneously equal to T_0

$$\frac{T}{T_0} = 1 - \frac{\pi^2 y_0^2}{8s_0^2} \left(\cos \frac{2\pi s}{s_0} + \frac{2\pi^2}{3} \cos 2\omega_0 t - \frac{2\pi^2 s^2}{s_0^2} \cos 2\omega_0 t \right)$$

and

$$\left(\frac{\Delta T}{T_0} \right)_{\max} \approx 9 \frac{y_0^2}{s_0^2}$$

This is the best operating condition of those cited, since it provides the greatest range of linear operation. However, a number of other possible choices which might be physically more attractive for some experiments produce essentially equivalent results.

⁸For free vibrations of the string mounted between fixed end supports

$$m \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s} \left\{ T_0 \left[1 + \frac{\Delta T(s,t)}{T_0} \right] \frac{\partial y}{\partial s} \right\} \quad \text{where } \frac{\Delta T}{T_0} \ll 1$$

If it is initially assumed that $\Delta T(s,t)$ can be neglected and we use a new independent variable (s/\hat{s}) to identify a particular element of the string (where the parameter \hat{s} is the instantaneous string length) then $y(s,t) = Y(s/\hat{s})G(t)$ and the solution to the above equation in the fundamental mode is

$$y(s,t) = y_0 \sin \frac{\pi s}{\hat{s}} \sin \omega_0 t \quad \text{where } \omega_0 = \frac{\pi}{X_0} \sqrt{\frac{T_0}{m}}$$

where X_0 is the initial string length. Now $y(x,t)$ can be obtained from arc length considerations and is given by

$$y(x,t) = y_0 \left(1 + \frac{\pi^2 y_0^2}{4X_0^2} \cos^2 \frac{\pi x}{X_0} \sin^2 \omega_0 t \right) \sin \frac{\pi x}{X_0} \sin \omega_0 t$$

From these equations and the equation of motion in the x -direction, we obtain the expression

$$\frac{T}{T_0} = 1 + \frac{\pi^2 y_0^2}{8X_0^2} \cos \frac{2\pi s}{\hat{s}} \cos 2\omega_0 t + \frac{\pi^2 y_0^2}{4X_0 \Delta L_0} \sin^2 \omega_0 t$$

where $\frac{\Delta L_0}{X_0}$ is the initial string strain. This approach is the first step in an iterative process which should yield more precise frequency and displacement equations applicable over large ranges of amplitude.

⁹In analyzing the behavior of tensioned strings we were concerned with the effect of damping on string motion, but it was not considered in the equations. The equations which result from considering a damping term are considerably more difficult to solve and their solution would have involved less valid

approximations than those which have been used in the analyses presented above. Fortunately, one of the most significant results of our experimental investigations was to demonstrate that the planar motion of the string is relatively unaltered by air damping. The string was driven at atmospheric pressure and at a pressure of 10^{-2} torr. These results are presented in Figs. 4 and 5 and indicate that no significant deviations were observed from the response predicted from Eqs. (16) and (23), which were derived without considering the effect of damping.

PARAMETRICALLY DRIVEN STRING WITH END DRIVER MOTION

$$-X_d \sin^2 \omega t$$

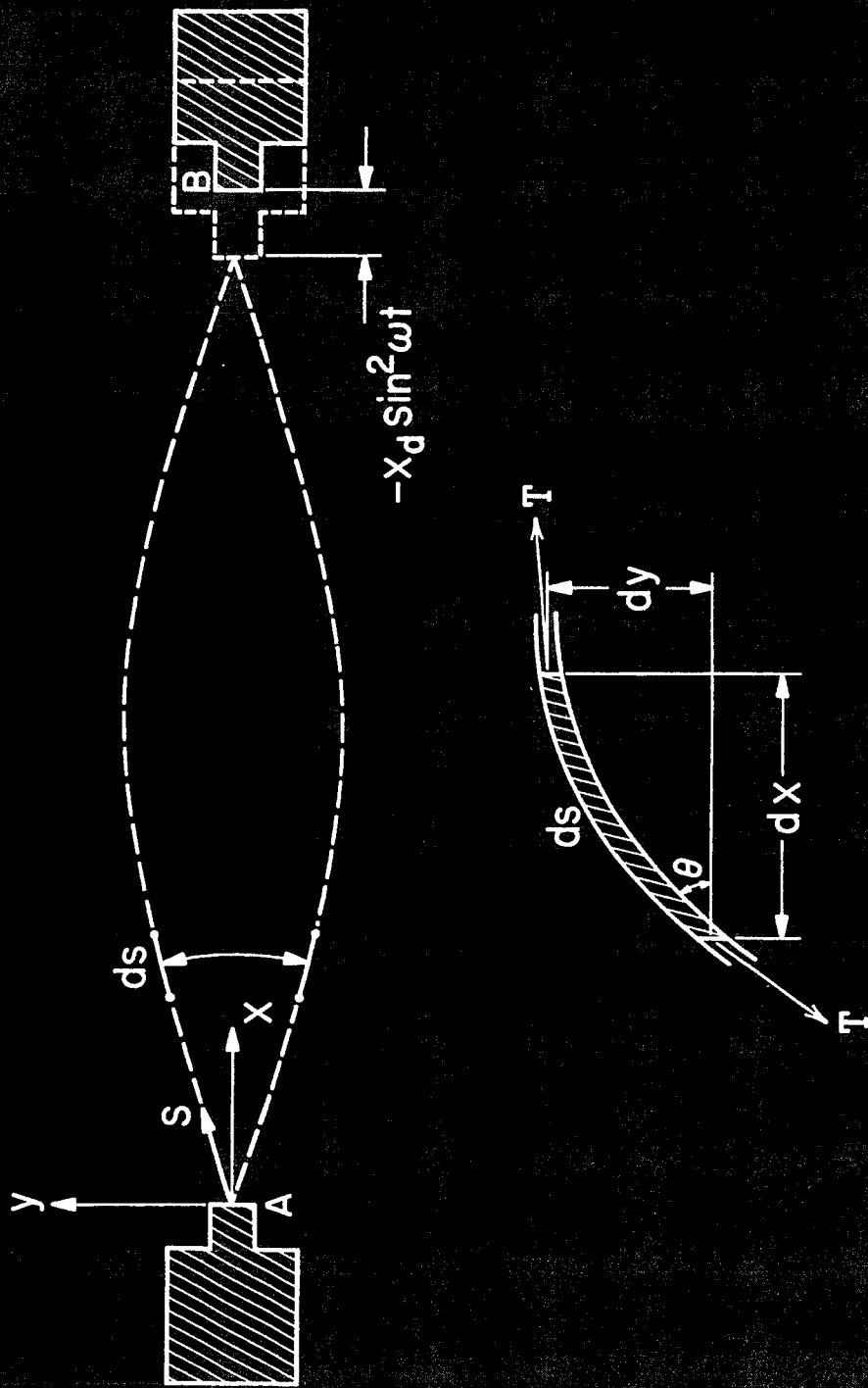


Fig 1.

STABILITY CRITERIA FOR THE VIBRATING STRING

DRIVER MOTION = $X_d' \cos 2\omega t$

- = "START UP" POINTS AT ATMOSPHERE
- ◇ = "START UP" POINTS AT VACUUM
- △ = "DROP OUT" POINTS AT VACUUM
- = "DROP OUT" POINTS AT ATMOSPHERE

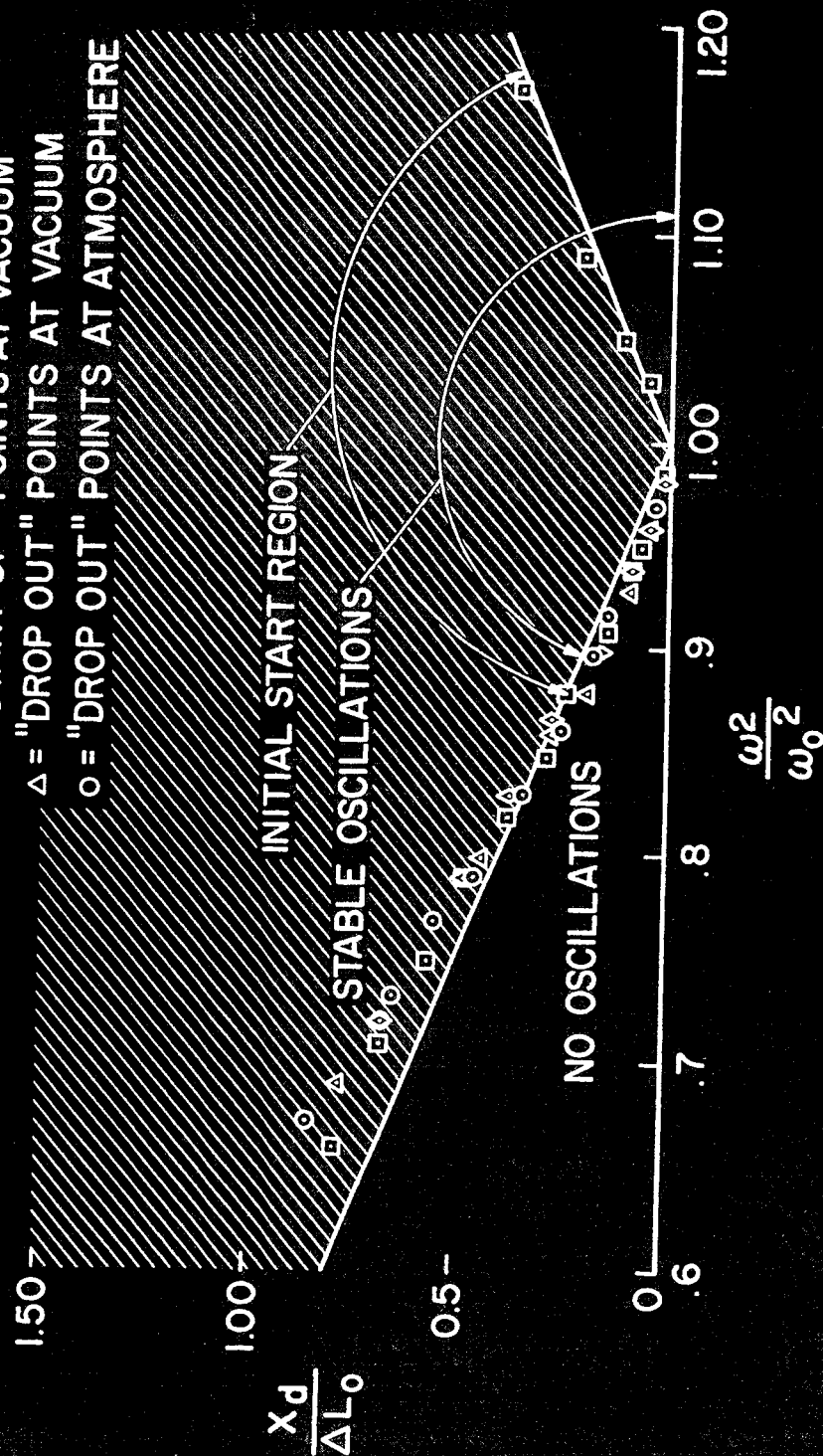


Fig 2

VIBRATING STRING APPARATUS

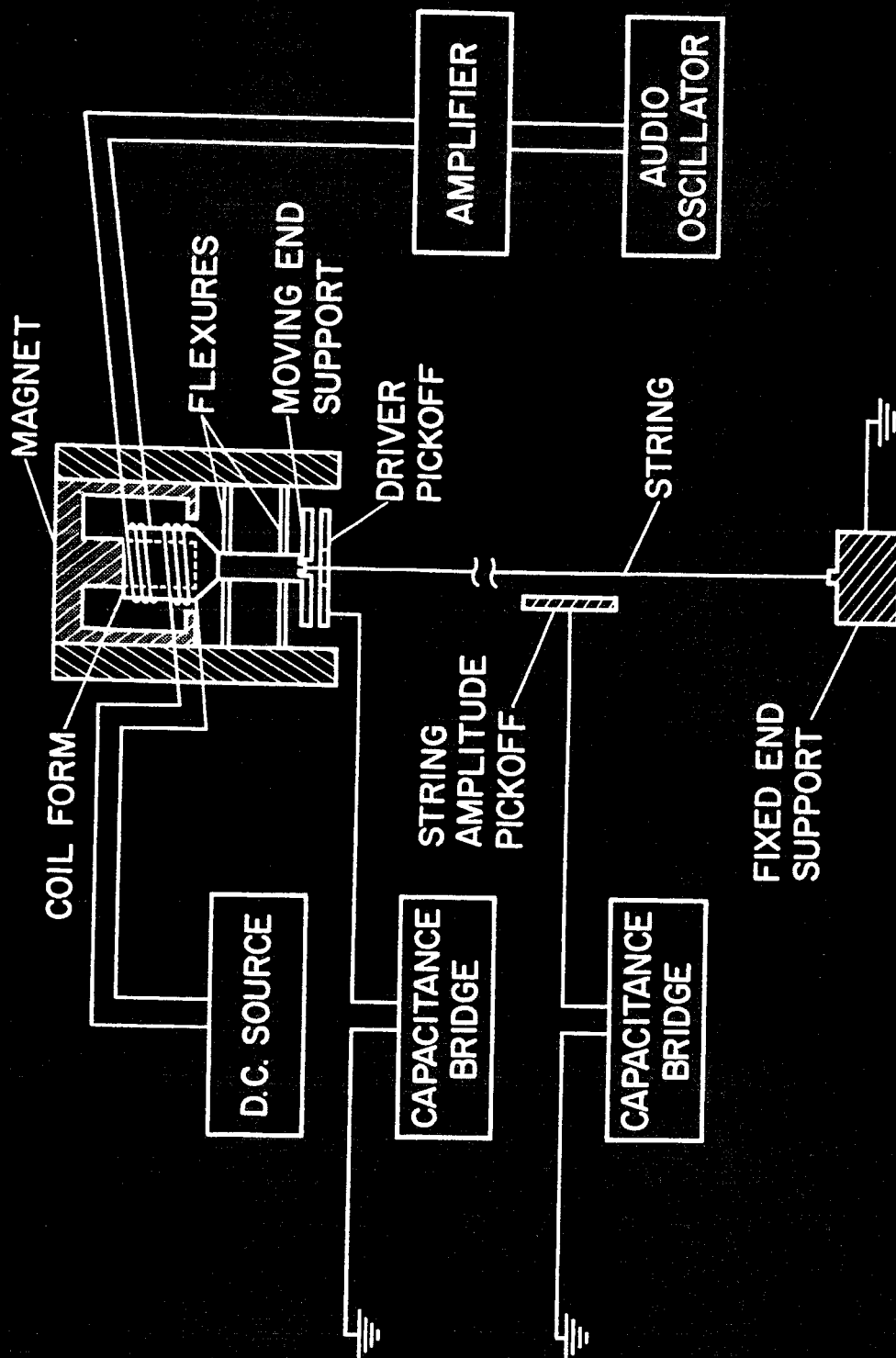


Fig. 3

FREQUENCY RESPONSE FOR THE END DRIVEN VIBRATING STRING

DRIVER MOTION $= -x_d \sin^2 \omega t$

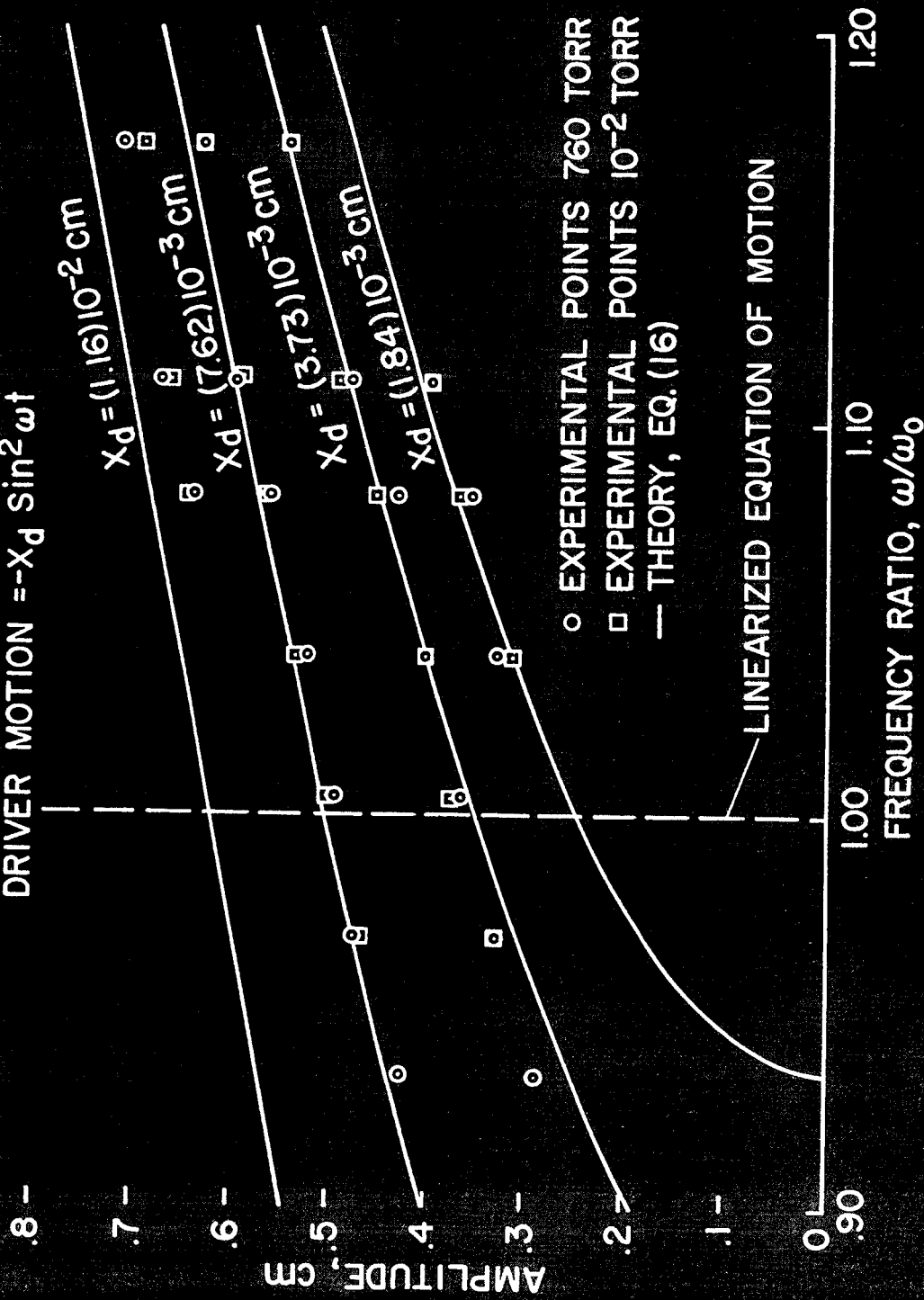


Fig 4

FREQUENCY RESPONSE FOR THE END DRIVEN VIBRATING STRING

$$\text{DRIVER MOTION} = X_d' \cos 2\omega t$$

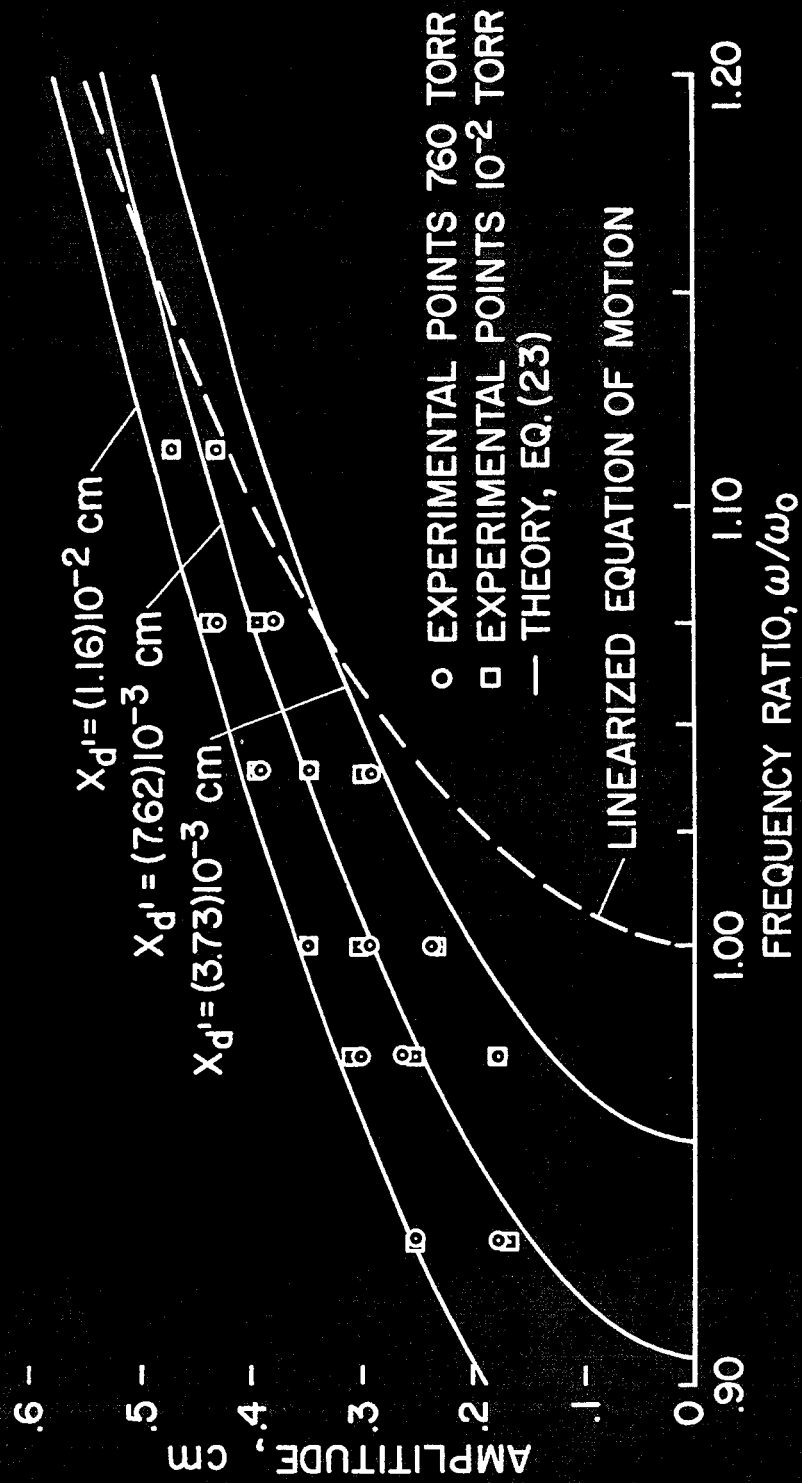


Fig 5

STABILITY CRITERIA FOR THE VIBRATING STRING DRIVER MOTION $= -X_d \sin^2 \omega t$

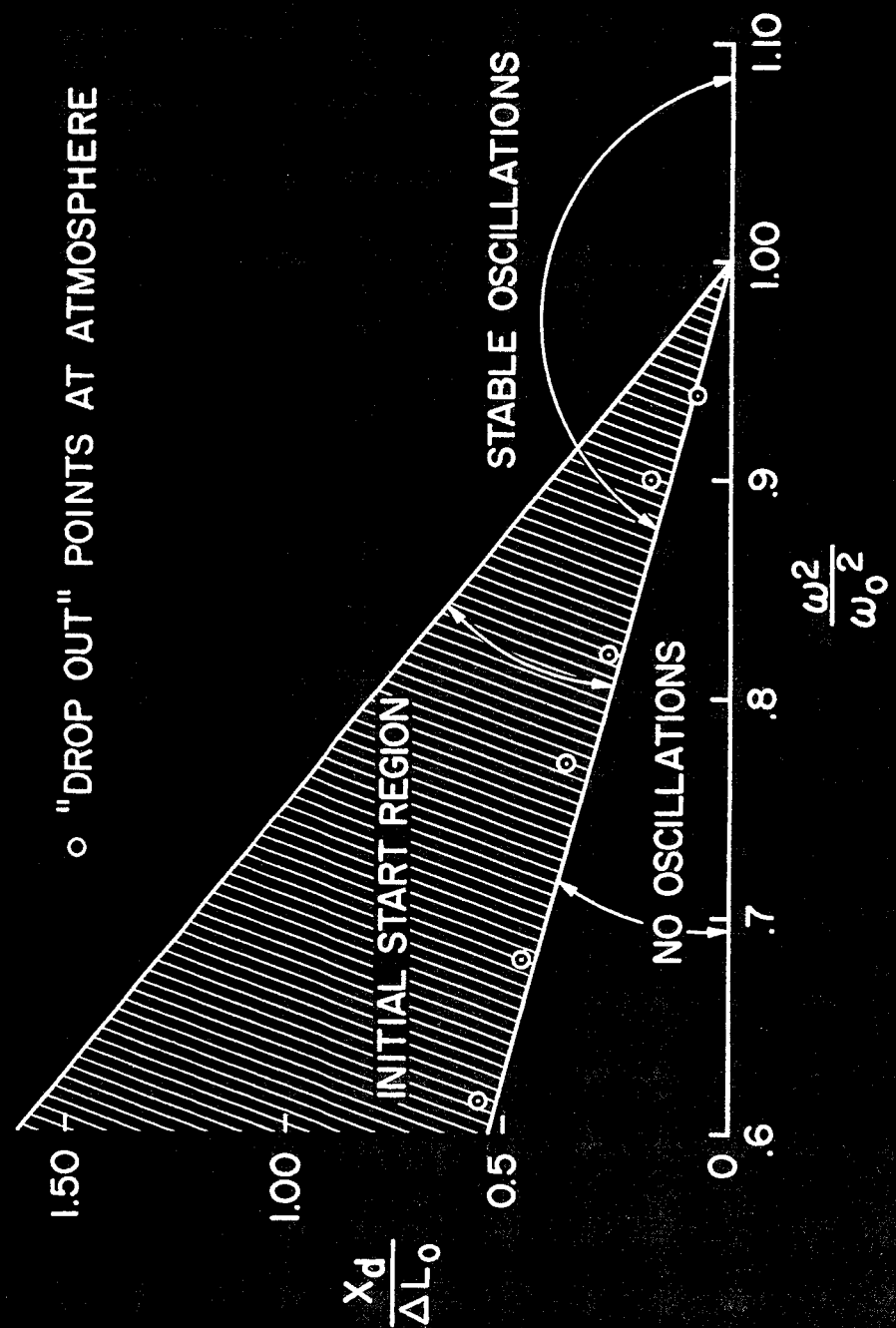


Fig. 6